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Q8

$A_{22} = (-1)^{2+2}\begin{vmatrix}1 & 0\\5 & -1\end{vmatrix} = -1$, $A_{23} = (-1)^{2+3}\begin{vmatrix}1 & 0\\5 & 2\end{vmatrix} = -2$

$A_{31} = (-1)^{3+1}\begin{vmatrix}0 & 0\\3 & 0\end{vmatrix} = 0$, $A_{32} = (-1)^{3+2}\begin{vmatrix}1 & 0\\3 & 0\end{vmatrix} = 0$

$A_{33} = (-1)^{3+3}\begin{vmatrix}1 & 0\\3 & 3\end{vmatrix} = 3$

$$\therefore \text{ adj } A = \begin{bmatrix}-3 & 3 & -9\\0 & -1 & -2\\0 & 0 & 3\end{bmatrix}' = \begin{bmatrix}-3 & 0 & 0\\3 & -1 & 0\\-9 & -2 & 3\end{bmatrix}$$

Hence, $A^{-1} = \dfrac{1}{|A|}(\text{adj } A) = -\dfrac{1}{3}\begin{bmatrix}-3 & 0 & 0\\3 & -1 & 0\\-9 & -2 & 3\end{bmatrix}$

9. $\begin{bmatrix}2 & 1 & 3\\4 & -1 & 0\\-7 & 2 & 1\end{bmatrix}$

Soln.: Let $A = \begin{bmatrix}2 & 1 & 3\\4 & -1 & 0\\-7 & 2 & 1\end{bmatrix}$

Then, $|A| = \begin{vmatrix}2 & 1 & 3\\4 & -1 & 0\\-7 & 2 & 1\end{vmatrix}$

$= 2(-1-0) - 1(4-0) + 3(8-7) = -2 - 4 + 3 = -3 \neq 0.$

So, A is a non-singular matrix and therefore, it is invertible. Let A_{ij} be cofactor of a_{ij} in A. Then, the cofactors of elements of A are given by

$A_{11} = (-1)^{1+1}\begin{vmatrix}-1 & 0\\2 & 1\end{vmatrix} = -1$, $A_{12} = (-1)^{1+2}\begin{vmatrix}4 & 0\\-7 & 1\end{vmatrix} = -4$

$A_{13} = (-1)^{1+3}\begin{vmatrix}4 & -1\\-7 & 2\end{vmatrix} = 8 - 7 = 1$

$A_{21} = (-1)^{2+1}\begin{vmatrix}1 & 3\\2 & 1\end{vmatrix} = -(1-6) = 5$

$A_{22} = (-1)^{2+2}\begin{vmatrix}2 & 3\\-7 & 1\end{vmatrix} = 2 + 21 = 23$

$A_{23} = (-1)^{2+3}\begin{vmatrix}2 & 1\\-7 & 2\end{vmatrix} = -(4+7) = -11$

$A_{31} = (-1)^{3+1}\begin{vmatrix}1 & 3\\-1 & 0\end{vmatrix} = 3$, $A_{32} = (-1)^{3+2}\begin{vmatrix}2 & 3\\4 & 0\end{vmatrix} = 12$

$A_{33} = (-1)^{3+3}\begin{vmatrix}2 & 1\\4 & -1\end{vmatrix} = -2 - 4 = -6$

$$\therefore \text{ adj } A = \begin{bmatrix}-1 & -4 & 1\\5 & 23 & -11\\3 & 12 & -6\end{bmatrix}' = \begin{bmatrix}-1 & 5 & 3\\-4 & 23 & 12\\1 & -11 & -6\end{bmatrix}$$

Hence, $A^{-1} = \dfrac{1}{|A|}(\text{adj } A) = -\dfrac{1}{3}\begin{bmatrix}-1 & 5 & 3\\-4 & 23 & 12\\1 & -11 & -6\end{bmatrix}$

10. $\begin{bmatrix}1 & -1 & 2\\0 & 2 & -3\\3 & -2 & 4\end{bmatrix}$

Soln.: Let $A = \begin{bmatrix}1 & -1 & 2\\0 & 2 & -3\\3 & -2 & 4\end{bmatrix}$

$\therefore\ |A| = 1(8-6) + 1(0+9) + 2(0-6)$

$= 2 + 9 - 12 = -1 \neq 0$

$A_{11} = (-1)^{1+1}\begin{vmatrix}2 & -3\\-2 & 4\end{vmatrix} = 8 - 6 = 2$

$A_{12} = (-1)^{1+2}\begin{vmatrix}0 & -3\\3 & 4\end{vmatrix} = -(0+9) = -9$

$A_{13} = (-1)^{1+3}\begin{vmatrix}0 & 2\\3 & -2\end{vmatrix} = 0 - 6 = -6$

$A_{21} = (-1)^{2+1}\begin{vmatrix}-1 & 2\\-2 & 4\end{vmatrix} = -(-4+4) = 0$

$A_{22} = (-1)^{2+2}\begin{vmatrix}1 & 2\\3 & 4\end{vmatrix} = 4 - 6 = -2$

$A_{23} = (-1)^{2+3}\begin{vmatrix}1 & -1\\3 & -2\end{vmatrix} = -(-2+3) = -1$

$A_{31} = (-1)^{3+1}\begin{vmatrix}-1 & 2\\2 & -3\end{vmatrix} = 3 - 4 = -1$

$A_{32} = (-1)^{3+2}\begin{vmatrix}1 & 2\\0 & -3\end{vmatrix} = -(-3-0) = 3$

$A_{33} = (-1)^{3+3}\begin{vmatrix}1 & -1\\0 & 2\end{vmatrix} = 2 + 0 = 2$

$$\therefore \text{ adj } A = \begin{bmatrix}2 & -9 & -6\\0 & -2 & -1\\-1 & 3 & 2\end{bmatrix}' = \begin{bmatrix}2 & 0 & -1\\-9 & -2 & 3\\-6 & -1 & 2\end{bmatrix}$$

Hence, $A^{-1} = \dfrac{\text{adj } A}{|A|} = \dfrac{1}{-1}\begin{bmatrix}2 & 0 & -1\\-9 & -2 & 3\\-6 & -1 & 2\end{bmatrix} = \begin{bmatrix}-2 & 0 & 1\\9 & 2 & -3\\6 & 1 & -2\end{bmatrix}$

11. $\begin{bmatrix}1 & 0 & 0\\0 & \cos\alpha & \sin\alpha\\0 & \sin\alpha & -\cos\alpha\end{bmatrix}$

Soln.: Let $A = \begin{bmatrix}1 & 0 & 0\\0 & \cos\alpha & \sin\alpha\\0 & \sin\alpha & -\cos\alpha\end{bmatrix}$, then

$|A| = 1(-\cos^2\alpha - \sin^2\alpha) = -1 \neq 0.$

$\therefore\ A^{-1}$ exists.

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So, A is non-singular matrix and therefore, A is invertible. Let A_{ij} be the cofactors of a_{ij} in A. Then the cofactors of elements of A are given by

For adjoint A, $A_{ij} = (-1)^{i+j} M_{ij}$

$$A_{11} = (-1)^{1+1}\begin{vmatrix}\cos\alpha & \sin\alpha\\ \sin\alpha & -\cos\alpha\end{vmatrix} = -\cos^2\alpha - \sin^2\alpha = -1$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix}0 & \sin\alpha\\ 0 & -\cos\alpha\end{vmatrix} = 0,\quad A_{13} = (-1)^{1+3}\begin{vmatrix}0 & \cos\alpha\\ 0 & \sin\alpha\end{vmatrix} = 0$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix}0 & 0\\ \sin\alpha & -\cos\alpha\end{vmatrix} = 0$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix}1 & 0\\ 0 & -\cos\alpha\end{vmatrix} = -\cos\alpha$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix}1 & 0\\ 0 & \sin\alpha\end{vmatrix} = -\sin\alpha$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix}0 & 0\\ \cos\alpha & \sin\alpha\end{vmatrix} = 0$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix}1 & 0\\ 0 & \sin\alpha\end{vmatrix} = -\sin\alpha$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix}1 & 0\\ 0 & \cos\alpha\end{vmatrix} = \cos\alpha$$

$$\therefore\quad \operatorname{adj} A = \begin{bmatrix}-1 & 0 & 0\\ 0 & -\cos\alpha & -\sin\alpha\\ 0 & -\sin\alpha & \cos\alpha\end{bmatrix}' = \begin{bmatrix}-1 & 0 & 0\\ 0 & -\cos\alpha & -\sin\alpha\\ 0 & -\sin\alpha & \cos\alpha\end{bmatrix}$$

Hence, $A^{-1} = \dfrac{1}{|A|}\operatorname{adj} A = \dfrac{1}{-1}\begin{bmatrix}-1 & 0 & 0\\ 0 & -\cos\alpha & -\sin\alpha\\ 0 & -\sin\alpha & \cos\alpha\end{bmatrix}$

$$= \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & \sin\alpha\\ 0 & \sin\alpha & -\cos\alpha\end{bmatrix}$$

12. Let $A = \begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}$ and $B = \begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Soln.: We have, $AB = \begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}\begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix} = \begin{bmatrix}67 & 87\\ 47 & 61\end{bmatrix}$

Since, $|AB| = 67\times 61 - 47\times 87 = -2 \neq 0$

So, AB is non-singular matrix and therefore,

$(AB)^{-1}$ exists and is given by $(AB)^{-1} = \dfrac{1}{|AB|}\operatorname{adj}(AB)$

$$= -\frac{1}{2}\begin{bmatrix}61 & -87\\ -47 & 67\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-61 & 87\\ 47 & -67\end{bmatrix}$$

Further, $|A| = 15 - 14 = 1 \neq 0$ and $|B| = 54 - 56 = -2 \neq 0$. So, A and B are both non-singular matrices and therefore, A^{-1} and B^{-1} both exist and are given by

$$A^{-1} = 1\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix},\ B^{-1} = -\frac{1}{2}\begin{bmatrix}9 & -8\\ -7 & 6\end{bmatrix}$$

$$\therefore\quad B^{-1}A^{-1} = -\frac{1}{2}\begin{bmatrix}9 & -8\\ -7 & 6\end{bmatrix}\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix}61 & -87\\ -47 & 67\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-61 & 87\\ 47 & -67\end{bmatrix}$$

Hence, $(AB)^{-1} = B^{-1}A^{-1}$.

13. If $A = \begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence, find A^{-1}.

Soln.: $A = \begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$

L.H.S. $= A^2 - 5A + 7I$

$$= \begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix} - 5\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix} + 7\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix} - \begin{bmatrix}15 & 5\\ -5 & 10\end{bmatrix} + \begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix}$$

$$= \begin{bmatrix}8-15+7 & 5-5+0\\ -5+5+0 & 3-10+7\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = O$$

$\Rightarrow\ A^2 - 5A + 7I = O$

Hence, proved.

Now, multiplying by A^{-1} on both sides, we get

$(A^{-1}A)A - 5AA^{-1} + 7IA^{-1} = O \ \Rightarrow\ IA - 5I + 7A^{-1} = O$

$\Rightarrow\ A - 5I + 7A^{-1} = O \ \Rightarrow\ 7A^{-1} = 5I - A$

$$\Rightarrow\ 7A^{-1} = 5\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix} \Rightarrow 7A^{-1} = \begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix} - \begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$$

$$\Rightarrow\ 7A^{-1} = \begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix} \ \Rightarrow\ A^{-1} = \frac{1}{7}\begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix}$$

14. For the matrix $A = \begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = 0$.

Soln.: We are given that, $A = \begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}$

Now, $A^2 + aA + bI = O$

$$\Rightarrow\ \begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix} + a\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix} + b\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = O$$

$$\Rightarrow\ \begin{bmatrix}9+2 & 6+2\\ 3+1 & 2+1\end{bmatrix} + \begin{bmatrix}3a & 2a\\ a & a\end{bmatrix} + \begin{bmatrix}b & 0\\ 0 & b\end{bmatrix} = O$$