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16. Solve the system of the following equations

$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4$, $\frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1$, $\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2$

Soln.: The equations can be written in the form $AX=B$,

where, $A=\begin{bmatrix}2 & 3 & 10\\ 4 & -6 & 5\\ 6 & 9 & -20\end{bmatrix}$, $X=\begin{bmatrix}1/x\\ 1/y\\ 1/z\end{bmatrix}$ and $B=\begin{bmatrix}4\\ 1\\ 2\end{bmatrix}$

Now, $|A|=\begin{vmatrix}2 & 3 & 10\\ 4 & -6 & 5\\ 6 & 9 & -20\end{vmatrix}$

$=2(120-45)-3(-80-30)+10(36+36)$

$=2(75)-3(-110)+10(72)=150+330+720=1200\neq 0$

$\therefore$ A^{-1} exists.

Now, let A_{ij} be the cofactor of the element in i^{th} row and j^{th} column. The cofactors are :

$A_{11}=(-1)^{1+1}\begin{vmatrix}-6 & 5\\ 9 & -20\end{vmatrix}=120-45=75,$

$A_{12}=(-1)^{1+2}\begin{vmatrix}4 & 5\\ 6 & -20\end{vmatrix}=-(-80-30)=110,$

$A_{13}=(-1)^{1+3}\begin{vmatrix}4 & -6\\ 6 & 9\end{vmatrix}=36+36=72,$

$A_{21}=(-1)^{2+1}\begin{vmatrix}3 & 10\\ 9 & -20\end{vmatrix}=-(-60-90)=150,$

$A_{22}=(-1)^{2+2}\begin{vmatrix}2 & 10\\ 6 & -20\end{vmatrix}=-40-60=-100,$

$A_{23}=(-1)^{2+3}\begin{vmatrix}2 & 3\\ 6 & 9\end{vmatrix}=-(18-18)=0,$

$A_{31}=(-1)^{3+1}\begin{vmatrix}3 & 10\\ -6 & 5\end{vmatrix}=15+60=75,$

$A_{32}=(-1)^{3+2}\begin{vmatrix}2 & 10\\ 4 & 5\end{vmatrix}=-(10-40)=30,$

$A_{33}=(-1)^{3+3}\begin{vmatrix}2 & 3\\ 4 & -6\end{vmatrix}=-12-12=-24$

$\therefore$ adj $A=\begin{bmatrix}75 & 110 & 72\\ 150 & -100 & 0\\ 75 & 30 & -24\end{bmatrix}'=\begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}$

Hence, $A^{-1}=\frac{1}{|A|}(\text{adj }A)=\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}$

As, $AX=B \Rightarrow X=A^{-1}B$

$\Rightarrow \begin{bmatrix}\frac{1}{x}\\ \frac{1}{y}\\ \frac{1}{z}\end{bmatrix}=\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}\begin{bmatrix}4\\ 1\\ 2\end{bmatrix}$

$=\frac{1}{1200}\begin{bmatrix}300+150+150\\ 440-100+60\\ 288+0-48\end{bmatrix}$

$\Rightarrow \begin{bmatrix}\frac{1}{x}\\ \frac{1}{y}\\ \frac{1}{z}\end{bmatrix}=\frac{1}{1200}\begin{bmatrix}600\\ 400\\ 240\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ \frac{1}{3}\\ \frac{1}{5}\end{bmatrix}$

Thus, $\frac{1}{x}=\frac{1}{2}, \frac{1}{y}=\frac{1}{3}, \frac{1}{z}=\frac{1}{5}$

Hence, $x=2, y=3, z=5$

Choose the correct answer in questions 17 to 19.

17. If a, b, c, are in A.P, then the determinant

$\begin{vmatrix}x+2 & x+3 & x+2a\\ x+3 & x+4 & x+2b\\ x+4 & x+5 & x+2c\end{vmatrix}$ is

(A) 0 **(B)** 1

(C) x **(D)** $2x$

Soln.: (A) : $\begin{vmatrix}x+2 & x+3 & x+2a\\ x+3 & x+4 & x+2b\\ x+4 & x+5 & x+2c\end{vmatrix}$

Applying $R_1 \rightarrow R_1 \quad R_2$, we get

$\begin{vmatrix}-1 & -1 & 2a-2b\\ x+3 & x+4 & x+2b\\ x+4 & x+5 & x+2c\end{vmatrix}$

Applying $C_1 \rightarrow C_1 - C_2$, we get

$\begin{vmatrix}0 & -1 & 2a-2b\\ -1 & x+4 & x+2b\\ -1 & x+5 & x+2c\end{vmatrix}$

Expanding along R_1, we get

$1\begin{vmatrix}-1 & x+2b\\ -1 & x+2c\end{vmatrix}+(2a-2b)\begin{vmatrix}-1 & x+4\\ -1 & x+5\end{vmatrix}$

$=-x-2c+x+2b+(2a-2b)[-x-5+x+4]$

$=2b-2c+(2a-2b)(-1)$

$=2b-2c-2a+2b=2[2b-(c+a)]$

$=2\left[2\left(\frac{a+c}{2}\right)-(c+a)\right]=0$ [Since a, b, c are in A.P.]

18. If x, y, z are non-zero real numbers, then the inverse of matrix $A=\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is

(A) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$ (B) $xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

(C) $\frac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ (D) $\frac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Soln.: (A) : Let $A=\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$.

$\therefore$ $|A| = xyz \neq 0$, A^{-1} exists.

Now, cofactors of elements of A are :

$A_{11}=(-1)^{1+1}\begin{vmatrix} y & 0 \\ 0 & z \end{vmatrix} = yz,$ $A_{12}=(-1)^{1+2}\begin{vmatrix} 0 & 0 \\ 0 & z \end{vmatrix} = 0,$

$A_{13}=(-1)^{1+3}\begin{vmatrix} 0 & y \\ 0 & 0 \end{vmatrix} = 0,$ $A_{21}=(-1)^{2+1}\begin{vmatrix} 0 & 0 \\ 0 & z \end{vmatrix} = 0,$

$A_{22}=(-1)^{2+2}\begin{vmatrix} x & 0 \\ 0 & z \end{vmatrix} = xz,$ $A_{23}=(-1)^{2+3}\begin{vmatrix} x & 0 \\ 0 & 0 \end{vmatrix} = 0,$

$A_{31}=(-1)^{3+1}\begin{vmatrix} 0 & 0 \\ y & 0 \end{vmatrix} = 0,$ $A_{32}=(-1)^{3+2}\begin{vmatrix} x & 0 \\ 0 & 0 \end{vmatrix} = 0,$

$A_{33}=(-1)^{3+3}\begin{vmatrix} x & 0 \\ 0 & y \end{vmatrix} = xy$

$\therefore$ $\text{adj } A = \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}' = \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$

Hence, $A^{-1} = \frac{1}{xyz}\begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix} = \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

19. Let $A=\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \leq \theta \leq 2\pi$. Then,

(A) Det $(A) = 0$ **(B)** Det $(A) \in (2, \infty)$

(C) Det $(A) \in (2, 4)$ **(D)** Det $(A) \in [2, 4]$

Soln. (D) : $\begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$

Expanding along R_1, we get

$$1(1 + \sin^2\theta) - \sin\theta(-\sin\theta + \sin\theta) + 1(\sin^2\theta + 1)$$
$$= 1 + \sin^2\theta + 1 + \sin^2\theta = 2(1 + \sin^2\theta)$$

As, $\sin^2\theta \in [0, 1]$

$\Rightarrow$ $1 + \sin^2\theta \in [1, 2]$

$\Rightarrow$ $2(1 + \sin^2\theta) \in [2, 4]$