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NCERT Section

Exercise - 2.1

Find the principal values of the following :

(1) $\sin^{-1}\left(-\frac{1}{2}\right)$ (2) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

(3) $\text{cosec}^{-1}(2)$ (4) $\tan^{-1}(-\sqrt{3})$

(5) $\cos^{-1}\left(-\frac{1}{2}\right)$ (6) $\tan^{-1}(-1)$

(7) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (8) $\cot^{-1}(\sqrt{3})$

(9) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ (10) $\text{cosec}^{-1}(-\sqrt{2})$

Soln.: (1) Let $\sin^{-1}\left(-\frac{1}{2}\right)=x \Rightarrow \sin x=-\frac{1}{2}$

We know that the range of the principal value branch of $\sin^{-1}$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Then, $\sin\left(-\frac{\pi}{6}\right)=-\frac{1}{2}$, where $-\frac{\pi}{6}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Hence, the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is $-\frac{\pi}{6}$

Soln.: (2) Let $\cos^{-1}\frac{\sqrt{3}}{2}=x \Rightarrow \frac{\sqrt{3}}{2}=\cos x$

We know that the range of the principal value branch of $\cos^{-1}$ is $[0,\pi]$

Then, $\cos x=\frac{\sqrt{3}}{2}=\cos\frac{\pi}{6}$, where $\frac{\pi}{6}\in[0,\pi]$

Hence, the principal value of $\cos^{-1}\frac{\sqrt{3}}{2}$ is $\frac{\pi}{6}$.

Soln.: (3) Let $\text{cosec}^{-1}(2)=x \Rightarrow 2=\text{cosec}\,x$

We know that the range of the principal value branch of cosec^{-1} is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$

Then, $2=\text{cosec}\,x=\text{cosec}\left(\frac{\pi}{6}\right)$, where $\frac{\pi}{6}\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]-\{0\}$

Hence, the principal value of $\text{cosec}^{-1}(2)$ is $\pi/6$.

Soln.: (4) Let $\tan^{-1}(-\sqrt{3})=x \Rightarrow -\sqrt{3}=\tan x$

We know that the range of the principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Then, $\tan x=-\sqrt{3}=\tan\left(-\frac{\pi}{3}\right)$, where $-\frac{\pi}{3}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Hence, the principal value of $\tan^{-1}(-\sqrt{3})$ is $-\frac{\pi}{3}$.

Soln.: (5) Let $x=\cos^{-1}\left(-\frac{1}{2}\right) \Rightarrow -\frac{1}{2}=\cos x$

We know that the range of principal value branch of $\cos^{-1}$ is $[0,\pi]$

Then, $\left(-\frac{1}{2}\right)=-\cos\frac{\pi}{3}=\cos\left(\pi-\frac{\pi}{3}\right)=\cos\left(\frac{2\pi}{3}\right)$,

where $\frac{2\pi}{3}\in[0,\pi]$

Hence, the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is $\frac{2\pi}{3}$.

Soln.: (6) Let $\tan^{-1}(-1)=x \Rightarrow -1=\tan x$

We know that the range of principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Then, $-1=\tan\left(-\frac{\pi}{4}\right)$ where $-\frac{\pi}{4}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Hence, the principal value of $\tan^{-1}(-1)$ is $-\frac{\pi}{4}$.

Soln.: (7) Let $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)=x \Rightarrow \sec x=\frac{2}{\sqrt{3}}$

We know that the range of principal value branch of $\sec^{-1}$ is $[0,\pi]-\left\{\frac{\pi}{2}\right\}$.

Then, $\frac{2}{\sqrt{3}}=\sec\left(\frac{\pi}{6}\right)$, where $\frac{\pi}{6}\in[0,\pi]-\left\{\frac{\pi}{2}\right\}$

Hence, the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is $\frac{\pi}{6}$.

Soln.: (8) Let $\cot^{-1}(\sqrt{3})=x \Rightarrow \sqrt{3}=\cot x$

We know that the range of principal value of $\cot^{-1}$ is $(0,\pi)$

Then, $\sqrt{3}=\cot\left(\frac{\pi}{6}\right)$, where $\frac{\pi}{6}\in(0,\pi)$

Hence, the principal value branch of $\cot^{-1}(\sqrt{3})$ is $\frac{\pi}{6}$.

Soln.: (9) Let $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)=x \Rightarrow -\frac{1}{\sqrt{2}}=\cos x$

We know that the range of principal value branch of $\cos^{-1}$ is $[0,\pi]$

Then, $-\frac{1}{\sqrt{2}}=-\cos\left(\frac{\pi}{4}\right)=\cos\left(\pi-\frac{\pi}{4}\right)=\cos\frac{3\pi}{4}$,

where $\frac{3\pi}{4}\in[0,\pi]$

Hence, the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is $\frac{3\pi}{4}$.

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7. (i) : $\sin^{-1}x+\sin^{-1}y=\begin{cases}\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}, & \text{if } -1\le x,y\le 1 \text{ and } x^2+y^2\le 1 \text{ or if } xy<0 \text{ and } x^2+y^2>1\\ \pi-\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}, & \text{if } 0<x,\ y\le 1 \text{ and } x^2+y^2>1\\ -\pi-\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}, & \text{if } -1\le x,y<0 \text{ and } x^2+y^2>1\end{cases}$

(ii) : $\sin^{-1}x-\sin^{-1}y=\begin{cases}\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}, & \text{if } -1\le x,y\le 1 \text{ and } x^2+y^2\le 1 \text{ or if } xy>0 \text{ and } x^2+y^2>1\\ \pi-\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}, & \text{if } 0<x\le 1, -1\le y\le 0 \text{ and } x^2+y^2>1\\ -\pi-\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}, & \text{if } -1\le x<0, 0<y\le 1 \text{ and } x^2+y^2>1\end{cases}$

8. (i) : $\cos^{-1}x+\cos^{-1}y=\begin{cases}\cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\}, & \text{if } -1\le x,y\le 1 \text{ and } x+y\ge 0\\ 2\pi-\cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\}, & \text{if } -1\le x,y\le 1 \text{ and } x+y\le 0\end{cases}$

(ii) : $\cos^{-1}x-\cos^{-1}y=\begin{cases}\cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\}, & \text{if } -1\le x,y\le 1 \text{ and } x\le y\\ -\cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\}, & \text{if } -1\le y\le 0, 0<x\le 1 \text{ and } x\ge y\end{cases}$

9. (i) : $2\sin^{-1}x=\begin{cases}\sin^{-1}\left(2x\sqrt{1-x^2}\right) & ,\text{if } -\frac{1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}\\ \pi-\sin^{-1}\left(2x\sqrt{1-x^2}\right) & ,\text{if } \frac{1}{\sqrt{2}}\le x\le 1\\ -\pi-\sin^{-1}\left(2x\sqrt{1-x^2}\right) & ,\text{if } -1\le x\le -\frac{1}{\sqrt{2}}\end{cases}$

(ii) : $3\sin^{-1}x=\begin{cases}\sin^{-1}(3x-4x^3) & ,\text{if } -\frac{1}{2}\le x\le\frac{1}{2}\\ \pi-\sin^{-1}(3x-4x^3) & ,\text{if } \frac{1}{2}<x\le 1\\ -\pi-\sin^{-1}(3x-4x^3) & ,\text{if } -1\le x<-\frac{1}{2}\end{cases}$

10. (i) : $2\cos^{-1}x=\begin{cases}\cos^{-1}(2x^2-1) & ,\text{if } 0\le x\le 1\\ 2\pi-\cos^{-1}(2x^2-1) & ,\text{if } -1\le x\le 0\end{cases}$

(ii) : $3\cos^{-1}x=\begin{cases}\cos^{-1}(4x^3-3x) & ,\text{if } \frac{1}{2}\le x\le 1\\ 2\pi-\cos^{-1}(4x^3-3x) & ,\text{if } -\frac{1}{2}\le x\le\frac{1}{2}\\ 2\pi+\cos^{-1}(4x^3-3x) & ,\text{if } -1\le x\le -\frac{1}{2}\end{cases}$

11. (i) : $2\tan^{-1}x=\begin{cases}\tan^{-1}\left(\frac{2x}{1-x^2}\right) & ,\text{if } -1<x<1\\ \pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right) & ,\text{if } x>1\\ -\pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right) & ,\text{if } x<-1\end{cases}$

(ii) : $3\tan^{-1}x=\begin{cases}\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) & ,\text{if } -\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}\\ \pi+\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) & ,\text{if } x>\frac{1}{\sqrt{3}}\\ -\pi+\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) & ,\text{if } x<-\frac{1}{\sqrt{3}}\end{cases}$

12. (i) : $2\tan^{-1}x=\begin{cases}\sin^{-1}\left(\frac{2x}{1+x^2}\right) & ,\text{if } -1\le x\le 1\\ \pi-\sin^{-1}\left(\frac{2x}{1+x^2}\right) & ,\text{if } x>1\\ -\pi-\sin^{-1}\left(\frac{2x}{1+x^2}\right) & ,\text{if } x<-1\end{cases}$

(ii) : $2\tan^{-1}x=\begin{cases}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) & ,\text{if } 0\le x<\infty\\ -\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) & ,\text{if } -\infty<x\le 0\end{cases}$

13. (i) : $\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$=\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)=\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)=\operatorname{cosec}^{-1}\frac{1}{x}$

(ii) : $\cos^{-1}x=\sin^{-1}\sqrt{1-x^2}=\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$=\cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)=\sec^{-1}\frac{1}{x}=\operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

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Table for the domain and range of inverse trigonometric functions

Functions	Domain	Range (principal value branch)
$y = \sin^{-1}x$	$[-1, 1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$y = \tan^{-1}x$	$(-\infty, \infty)$	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$y = \text{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$y = \sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$y = \cot^{-1}x$	$(-\infty, \infty)$	$(0, \pi)$

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

1. (i) $\sin^{-1}(\sin x) = x, \quad \forall x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

   (ii) $\cos^{-1}(\cos x) = x, \quad \forall x \in [0, \pi]$

   (iii) $\tan^{-1}(\tan x) = x, \quad \forall x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

   (iv) $\text{cosec}^{-1}(\text{cosec}\, x) = x, \quad \forall x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right], x \neq 0$

   (v) $\sec^{-1}(\sec x) = x, \quad \forall x \in [0, \pi], x \neq \frac{\pi}{2}$

   (vi) $\cot^{-1}(\cot x) = x, \quad \forall x \in (0, \pi)$

2. (i) $\sin(\sin^{-1}x) = x, \quad \forall x \in [-1, 1]$

   (ii) $\cos(\cos^{-1}x) = x, \quad \forall x \in [-1, 1]$

   (iii) $\tan(\tan^{-1}x) = x, \quad \forall x \in (-\infty, \infty)$

   (iv) $\text{cosec}(\text{cosec}^{-1}x) = x, \quad \forall x \in (-\infty, -1] \cup [1, \infty)$

   (v) $\sec(\sec^{-1}x) = x, \quad \forall x \in (-\infty, -1] \cup [1, \infty)$

   (vi) $\cot(\cot^{-1}x) = x, \quad \forall x \in (-\infty, \infty)$

3. (i) $\sin^{-1}(-x) = -\sin^{-1}x, \quad \forall x \in [-1, 1]$

   (ii) $\cos^{-1}(-x) = \pi - \cos^{-1}x, \quad \forall x \in [-1, 1]$

   (iii) $\tan^{-1}(-x) = -\tan^{-1}x, \quad \forall x \in (-\infty, \infty)$

   (iv) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x, \quad \forall x \in (-\infty, -1] \cup [1, \infty)$

   (v) $\sec^{-1}(-x) = \pi - \sec^{-1}x, \quad \forall x \in (-\infty, -1] \cup [1, \infty)$

   (vi) $\cot^{-1}(-x) = \pi - \cot^{-1}x, \quad \forall x \in (-\infty, \infty)$

4. (i) $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1}x, \quad \forall x \in (-\infty, -1] \cup [1, \infty)$

   (ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x, \quad \forall x \in (-\infty, -1] \cup [1, \infty)$

   (iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x, \text{ for } x > 0 \\ -\pi + \cot^{-1}x, \text{ for } x < 0 \end{cases}$

5. (i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \quad \forall x \in [-1, 1]$

   (ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \quad \forall x \in (-\infty, \infty)$

   (iii) $\text{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}, \quad \forall x \in (-\infty, 1] \cup [1, \infty)$

6. (i) $\tan^{-1}x + \tan^{-1}y$

$$= \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } xy < 1 \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } x > 0, y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$$

   (ii) $\tan^{-1}x - \tan^{-1}y$

$$= \begin{cases} \tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{if } xy > -1 \\ \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{if } x > 0, y < 0 \text{ and } xy < -1 \\ -\pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{if } x < 0, y > 0 \text{ and } xy < -1 \end{cases}$$

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(iii) : $\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

$= \cot^{-1}\frac{1}{x} = \sec^{-1}\sqrt{1+x^2} = \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$

Note : Important substitutions to simplify trigonometric expressions involving inverse trigonometric functions.

(i) For $\sqrt{a^2 - x^2}$; we substitute $x = a\sin\theta$ or $x = a\cos\theta$

(ii) For $\sqrt{a^2 + x^2}$; we substitute $x = a\tan\theta$ or $x = a\cot\theta$

(iii) For $\sqrt{x^2 - a^2}$; we substitute $x = a\sec\theta$ or $x = a\operatorname{cosec}\theta$

(iv) For $\sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}}$; we substitute $x = a\cos\theta$ or $x = a\cos 2\theta$.